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A new integrable gravitational billiard

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Abstract. We discuss gravitational billiards, i.e. the two-dimensional motion of a point mass inside a hard boundary curve under the influence of a constant (e.g. gravitational) field. A parabolic boundary is shown to be a new example of integrable billiards. The system has a second integral of motion in addition to the energy, which is constructed analytically. Stability and bifurcation properties of the central periodic orbits are discussed. the results also shed new light on the known integrable case of elliptic non-gravitational billiards.

1. Introduction

The dynamics of Hamiltonian systems is nicely illustrated by the so-called billiard systems: a point mass reflected elastically from a boundary curve in a two-dimensional plane. Such billiard systems have often been studied (e.g. see Arvieu and Ayant 1987, Arvieu *et al* 1987, Ayant and Arvieu 1987, Benettin and Strelcyn 1978, Berry 1981, 1983, Berry and Robnik 1986, Crespi *et al* 1989, Keller and Rubinov 1960, Korsch and Jodl 1991, Korsch *et al* 1987, Lehtihet and Miller 1986, Mirbach and Korsch 1989, Poritzky 1950, Richter *et al* 1990, Robnik 1983, 1986, Saito *et al* 1982) because they offer various advantages over motion in continuous potential fields: the dynamics reduces in an obvious way to an iterated map between successive collisions with the boundary. This mapping can be quite easily constructed numerically or even analytically and it is not necessary to integrate differential equations and the discrete Poincaré maps appear almost naturally.

Ordinary billiards

The dynamics of a particle moving freely on a two-dimensional plane billiard table bounded by a closed convex boundary curve has been well studied (e.g. see Berry 1981, Korsch and Jodl 1991, Korsch *et al* 1987; references to the special case of polygonal billiards can be found in the papers by Berry (1981) and Mirbach and Korsch (1989)). Smooth billiards (i.e. bounded by a curve with a continuously turning tangent) show the generic behaviour of mixed regular and chaotic dynamics. The stadium billiard has been proven to be ergodic (Bunimovich 1974, 1979; see also Berry 1981). A single smooth billiard has been shown to be integrable: the elliptic one (for any value of the eccentricity). A second constant of motion has been explicitly constructed by Berry (1981), which can be rewritten as the product of the angular momenta about the two focal points of the ellipse (this interpretation is due to

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J H Hannay, as cited by Berry (1981) (see also Arvieu *et al* 1987); a simple proof can be found in Korsch and Jodl (1991) and Korsch *et al* (1987)). Strong evidence has been obtained by Poritzky (1950) supporting the conclusion that the elliptic boundary is the *only* integrable case. A generic distortion of the elliptic billiard leads to generic chaotic dynamics. A theorem by Lazutkin (1973) proves that billiards with sufficiently smooth (C^{553}) boundaries possess orbits which envelop caustics, which means that these billiards cannot be ergodic (see also Berry 1983).

Gravitational billiards

The dynamics of a falling body in a symmetric wedge under the influence of a constant force has been studied in detail by Lehtihet and Miller (1986) (see also the available computer programs for an interactive study of these system by Miller and Lehtihet (1990) or Korsch and Jodl (1991)). This system constitutes the prototype of a gravitational billiard, described by the Hamiltonian (mass = 1)

$$H = \tfrac{1}{2}(p_x^2 + p_y^2) + gy \tag{1}$$

with $g \geqslant 0$ and a boundary curve

$$y = f(x) \tag{2}$$

which restricts the motion of the particle to the region $y \geqslant f(x)$. At the boundary we have elastic reflection. In contrast to the ordinary billiards discussed above, the boundary is usually not closed. The symmetric wedge is given by the boundary

$$f(x) = b|x|. \tag{3}$$

In the limits $b \to 0$ and $b \to \infty$ the wedge billiard is integrable. Another integrable case occurs for $b = 1$, where the system is separable in Cartesian coordinates orthogonal to the wedge. For angles $1 < b < \infty$ the dynamics is generic, i.e. we find KAM-like behaviour, which undergoes fascinating bifurcations when b is varied. For any b in $0 < b < 1$ the dynamics seems to be completely chaotic, suggesting K-system behaviour (Lehtihet and Miller 1986). The untypical embedding of the integrable case at the boundary between the generic and the completely chaotic regime is due to the non-differentiability of the boundary at $x = 0$.

2. Smooth gravitational billiards

The wedge gravitational billiard offers little flexibility because it depends only on a single parameter, b. The other parameters (the force g and the energy E) can be removed by a scaling transformation. Here we discuss billiards with a smooth (C^1) boundary $f(x)$. In particular we choose $f(x) \leqslant f(0) = 0$ and $f' = (0) = 0$.

In most simple terms the dynamics can be described by the bounce mapping

$$(p_x', p_y', x', y') \xleftarrow{B} (p_x, p_y, x, y). \tag{4}$$

Here (x, y) denote the coordinates of the collision point, (p_x, p_y) are the components of the momentum immediately after the collision with the boundary and the primed quantities are the data of the subsequent collision with the boundary. The mapping B can be decomposed as

$$B = R \circ C \tag{5}$$

where the collision map C maps a collision point with the boundary onto the next one. R describes the elastic reflection at the boundary curve.

The collision map C is given by the next intersection of the parabolic trajectory of the particle in the 'gravitational' field

$$y(x') = y + \frac{p_y(x'-x)}{p_x} - \frac{g(x'-x)^2}{(2p_x^2)} \tag{6}$$

with the boundary curve $f(x)$. The (numerical) solution of

$$y(x') = f(x') = y' \tag{7}$$

determines the next collision point (x', y'). The momenta immediately before the collision with the boundary are given by

$$\begin{aligned} p_x'' &= p_x \\ p_y'' &= p_y - \frac{g(x'-x)}{p_x}. \end{aligned} \tag{8}$$

For the case of the wedge boundary, equation (3), the collision map can be given in closed form (Lehtihet and Miller 1986). The same can be achieved for the 'parabolic well'

$$f(x) = \tfrac{1}{2}ax^2. \tag{9}$$

A short calculation gives

$$x' = \frac{(g - ap_x^2)x + 2p_x p_y}{g + ap_x^2}. \tag{10}$$

The reflection map R can be most easily stated in terms of the tangential and normal momenta

$$\begin{pmatrix} p_t \\ p_n \end{pmatrix} = N \begin{pmatrix} 1 & f' \\ -f' & 1 \end{pmatrix} \begin{pmatrix} p_x \\ p_y \end{pmatrix} \tag{11}$$

with $N = (1 + f'^2)^{-1/2}$ and $f' = (\mathrm{d}f/\mathrm{d}x)(x)$. The reflection R at point (x', y'), $p_t \to p_t$, $p_n \to -p_n$, gives (rewritten in terms of the momenta p_x, p_y)

$$(p_x', p_y') \xleftarrow{R} (p_x'', p_y'') \tag{12a}$$

$$\begin{pmatrix} p_x' \\ p_y' \end{pmatrix} = N_+^{-1} \begin{pmatrix} N_- & 2f' \\ 2f' & -N_- \end{pmatrix} \begin{pmatrix} p_x'' \\ p_y'' \end{pmatrix} \tag{12b}$$

with $N_\pm = 1 \pm f'^2$ with $f' = \mathrm{d}f/\mathrm{d}x(x')$. Equations (7), (8) and (12) define the bounce mapping B.

The dynamics can be completely recorded by the Poincaré map M, which relates the x coordinates and the tangential momenta at subsequent collisions with the boundary

$$\begin{pmatrix} x^{n+1} \\ p_t^{n+1} \end{pmatrix} = M \begin{pmatrix} x^n \\ p_t^n \end{pmatrix}. \tag{13}$$

The y values can be reconstructed from equation (2), and p_n is given by conservation of energy (note that p_n is always positive); p_x and p_y are then determined by the inverse of the rotation (11). Figure 1 shows an example of the iterated map (13) for the boundary $f(x) = \frac{1}{2}ax^2 + \frac{1}{4}dx^4$ ($g = 1$, energy $E = 20$) with a distorted parabolic boundary

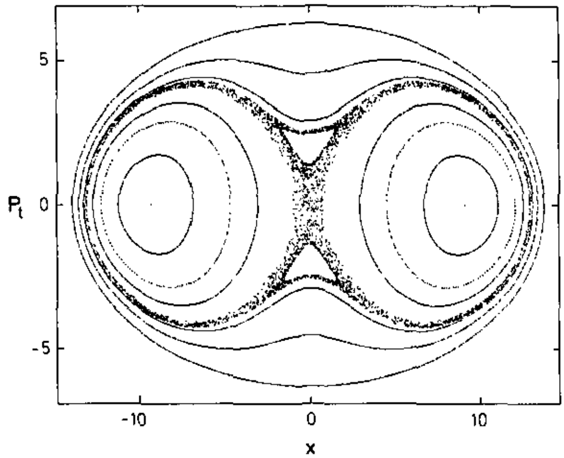

Figure 1. Poincaré map M for the gravitational billiard $f(x)=ax^2/2+dx^4/4$ ($a=0.2$, $d=4\times10^{-5}$, $g=1$, $E=20$), showing generic KAM-like behaviour.

($a=0.1$, $d=10^{-5}$), displaying generic KAM-like behaviour. We note that energy conservation restricts the (x, p_t): values to the region

$$\tfrac{1}{2}p_t^2+gf(x)\leqslant E. \tag{14}$$

For the parabolic billiard (9) this is the interior of the ellipse

$$p_t^2+gax^2=2E. \tag{15}$$

Figures 2(a–c) show the action of the Poincaré map M for the parabolic billiard (9) (again we choose $g=1$ and energy $E=20$) for three values of the parameter a: 0.001 (a), 0.02 (b), 0.1 (c). We note that all points seem to lie on invariant curves (even after the central fixed point has become unstable), strongly suggesting integrability of the system.

Because of $f'(0)=0$ the gravitational billiards have a central periodic orbit starting vertically at $x=0$, i.e. $(x, p_t)=(0,0)$ is a fixed point of the mapping M. The stability of this orbit can be easily discussed: linearization of M in the vicinity of $(0,0)$ gives

$$\begin{pmatrix} x' \\ p_t' \end{pmatrix} = \begin{pmatrix} \dfrac{1-4h}{R} & 2\delta \\ -\dfrac{4h}{\delta R}\left(1-\dfrac{2h}{R}\right) & 1-\dfrac{4h}{R} \end{pmatrix} \begin{pmatrix} x \\ p_t \end{pmatrix} \tag{16}$$

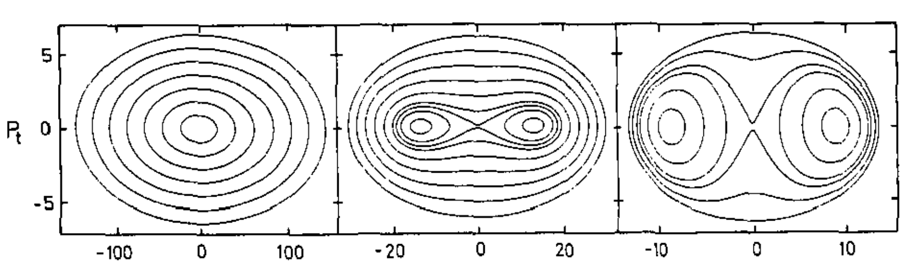

Figure 2. Poincaré map M for the gravitational parabolic billiard $f(x)=ax^2/2$ for $g=1$, $E=20$ and three values of a: 0.002 (a), 0.04 (b) and 0.2 (c). The dynamics is integrable.

where we have introduced the radius of curvature $R = 1/f''(0)$ of the boundary curve at $x=0$, the maximum height $h = E/g$ and $\delta^2 = 2h/g$ (compare also Berry 1981, equation (19) for the ordinary billiard). The stability of the fixed point is determined by the eigenvalues of the linearized mapping M_L, where stability is associated with complex eigenvalues on the unit circle, i.e.

$$|\mathrm{Tr}\, M_L| = \begin{cases} 2|1-(4h/R)| < 2 & \text{(stable)} \\ 2|1-(4h/R)| > 2 & \text{(unstable)} \end{cases} \tag{17}$$

(note that $\det M_L = 1$). Hence the vertical bounce orbit is unstable for heights exceeding the critical height $h_c = R/2$. Note that for the case of the parabolic billiard (9) the critical point $(0, h_c) = (0, 1/2a)$ is identical to the focal point of the parabola. For $h > h_c$ (i.e. $E > gh_c$) a stable orbit (a parabola) appears, which encloses the focal point (see figure 3). This new periodic orbit remains stable for all energies, whereas for billiards deviating from the parabolic form stability is lost with increasing energy. For the case shown in figures 2(a–c) ($g=1$, $E=20$, various values of a) we find a critical value of $a_c = 0.025$ in agreement with the numerical observations.

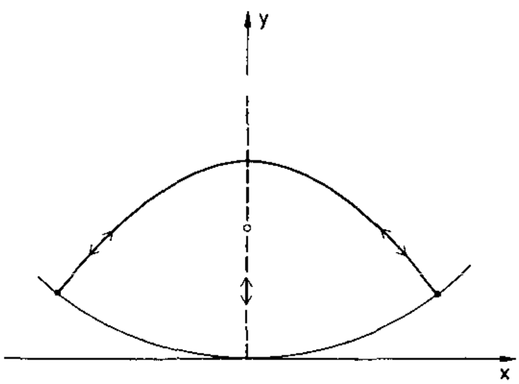

Figure 3. Stable (——) and unstable (– – –) periodic orbits for $h > h_c$. The dot marks the focal point.

3. An integrable billiard

The parabolic gravitational billiard is indeed a second example of a smooth integrable billiard. This can be easily proved by constructing the second integral of motion. We first observe that the integral must be invariant with respect to both the collision map C and the reflection map R.

It is well known that a superposition of a central c/r field and a constant field in the y direction

$$H_c = H + c/r \tag{18}$$

has an additional constant of motion (a generalization of the Lenz–Runge vector for the pure c/r field). Choosing the centre of the c/r field at $(0, y_0)$ on the y axis, this constant of motion reads

$$G = -\left(\frac{c}{r} + p_x(zp_x - xp_y)\right) + \tfrac{1}{2}gx^2 \tag{19}$$

(Landau and Lifshitz 1976, Helfrich 1972), where $z = y - y_0$ is the distance from the centre. In the following it will be shown that for the case $y_0 = 1/2a$, i.e. when the centre of the force coincides with the focal point of the boundary parabola, the parabolic billiard will be integrable. It remains to be shown that the integral G of the free gravitational motion is also an invariant of the reflection map R. First we note that during the action of R all position coordinates remain constant. Hence the only part of interest is the quantitity

$$\tau = p_x(zp_x - xp_y). \tag{20}$$

It is convenient to rewrite τ in terms of the tangential and normal momenta (see equation (11)) as

$$\tau = N^2\{(z - xf')p_t^2 + f'(zf' + x)p_n^2 - [2zf' + x(1 - f'^2)]p_t p_n\}. \tag{21}$$

With $f' = ax$ and $z = y - 1/2a = \frac{1}{2}ax^2 - 1/2a$ we find

$$2zf' + x(1 - f'^2) = 0 \tag{22}$$

and hence

$$\tau = \frac{-p_t^2 + a^2x^2p_n^2}{2a} \tag{23}$$

which is clearly unchanged by the reflection $p_t \to p_t$, $p_n \to -p_n$.

Conversely equation (22) reads, after introducing $z(x) = y - y_0 = f(x) - y_0$,

$$xz'^2 = 2zz' - x = 0. \tag{24}$$

This differential equation is well known (Kamke 1977) and has the unique solution

$$z(x) = \tfrac{1}{2}ax^2 - \frac{1}{2a} \tag{25}$$

i.e. the parabolic billiard (9) is the only integrable billiard with integral (19). It is still an open question, however, whether the parabolic billiard is the only existing gravitational billiard with a smooth boundary.

Note that for $c = 0$ we have exactly the parabolic gravitational billiard above.

The contour lines of the invariant G at fixed energy

$$E = \frac{p_t^2 + p_n^2}{2 + gy} \tag{26}$$

(assuming a case without a c/r field), i.e.

$$G = -\tau + \tfrac{1}{2}gx^2 = \frac{(1 + a^2x^2)p_t^2}{2a} + (\tfrac{1}{2}g - aE)x^2 + \tfrac{1}{2}ga^2x^4 \tag{27}$$

are in agreement with the numerically calculated invariant curves shown in figures 2(a–c). We can also derive the periodic orbits (the fixed points of the iterated map M^k) from the extremal points of $G(x, p_t)$: for $E < g/2a$ $G(x, p_t)$ has a single minimum at the origin ($x = 0$, $p_t = 0$), which turns into a saddlepoint for $E > g/2a$ and two new minima appear at $(x, p_t) = (\pm x_m, 0)$ with $x_m = (aE/g - \frac{1}{2})^{1/2}/a$. The minima of G are stable fixed points, the saddlepoint is unstable. These are the only fixed points for the parabolic billiard.

4. Concluding remarks

It has been shown that the parabolic gravitational billiard (with an additional Coulomb field centred in the focal point of the boundary parabola) is integrable. This is directly related to separability of the problem in parabolic coordinates (see e.g.) Landau and Lifshitz 1976, Helfrich 1972). It is now interesting to note that use of the closely related elliptic coordinates (e.g. as discussed in Morse and Feshbach 1953) directly leads to separability of the (non-gravitational) elliptic billiard

$$\left(\frac{x}{A}\right)^2+\left(\frac{y}{B}\right)^2=1 \tag{28}$$

which is therefore integrable (see introduction). A second integral of motion is the product of angular momenta about the two focal points (Berry 1981, Arvieu *et al* 1987, Korsch and Jodl, 1991, Korsch *et al* 1987):

$$F=L_1L_2=[(x+e)\,p_y-yp_x][(x-e)\,p_y-yp_x] \tag{29}$$

where $e=(A^2-B^2)^{1/2}$ is the distance of the focal points from the centre. It immediately follows that F can be rewritten as

$$F=y^2p_y^2-L^2 \tag{30}$$

where $L=xp_y-yp_x$ is the angular momentum about the centre of the ellipse. This is identical to the well-known invariant (Erikson and Hill 1949, Helfrich 1972, Landau and Lifshitz 1976)

$$F=y^2p_y^2-L^2+2my(c_1\cos(\theta_1)+c_2\cos(\theta_2)) \tag{31}$$

for the two-centre Coulomb field

$$V(x,y)=\frac{c_1}{r_1}+\frac{c_2}{r_2}. \tag{32}$$

Here r_1 and r_2 are the distances from the two centres, which are placed symmetrically at $\pm e$ on the x axis and θ_1, θ_2 are the angles between the focal rays and the x axis. The case $c_1=c_2=0$ reduces to the ordinary elliptic billiard.

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